Automatic Control Systems

Part II:

Laplace Transform and Time-Domain Analysis

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IV. Laplace Transform:

The Laplace transform is one of the mathematical tools used for the solution of ordinary linear differential equations. The Laplace transform method has the following two attractive features:

1. The homogeneous equation and the particular integral are solved in one operation.
2. The Laplace transform converts the differential equation into an algebraic equation in s. It is possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the s-domain. The final solution is obtained by taking the inverse Laplace transform.

Definition of the Laplace Transform:

Given the function $f(t)$ that satisfies the condition:

$$\int_0^\infty \left| f(t)e^{-\sigma t} \right| dt < \infty, \tag{4-1}$$

for some finite real σ, the Laplace transform of $f(t)$ is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \tag{4-2}$$

or

$$F(s) = \mathcal{L}[f(t)]. \tag{4-3}$$

The variable s is referred to as the Laplace operator, which is a complex variable, $s = \sigma + j\omega$, where σ is the real part and ω is the imaginary part, as shown in Figure 4-1.

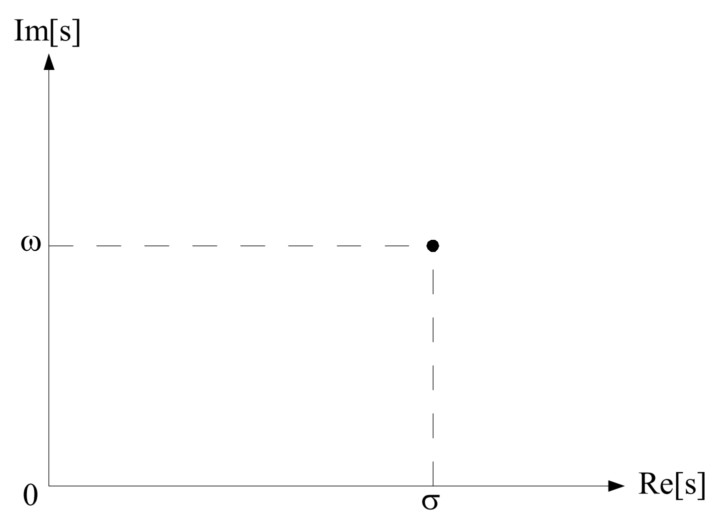


Fig. 4-1. Graphical presentation of Laplace operator (s-Plane).

Example 4-1: Let $f(t)$ be a unit step function that is defined to have a constant value of unity for $t \geq 0$ and a zero for $t < 0$, namely, $f(t) = u_s(t)$. What is its Laplace transform F(s)?
Solution:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty u_s(t)e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s} \quad \blacklozenge$$

Example 4-2: Let f(t) is an exponential function, $f(t) = e^{-at}$ $t \geq 0$, where a is a constant. What is its Laplace transform F(s)?
Solution:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}\, e^{-st}dt = \int_0^\infty e^{-(s+a)t}\, dt = -\frac{1}{(s+a)}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a} \quad \blacklozenge$$

Important Theorems of the Laplace Transform:

1. Multiplication by a constant-

$$\mathcal{L}[kf(t)] = kF(s) \tag{4-4}$$

2. Sum and difference-

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \tag{4-5}$$

3. Differentiation-

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t\to\infty} f(t) = sF(s) - f(0) \tag{4-6}$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0) \tag{4-7}$$

where $f^{(k)}(0) = \frac{d^k f(t)}{dt^k}\Big|_{t=0}$.

4. Integration-

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s} \tag{4-8}$$

$$\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}}\int_0^{t_n} f(\tau)d\tau\, dt_1\, dt_2 \cdots dt_{n-2}\, dt_{n-1}\right] = \frac{F(s)}{s^n} \tag{4-9}$$

5. Shift in time-

$$\mathcal{L}[f(t - T)u_s(t - T)] = e^{-Ts}F(s) \tag{4-10}$$

6. Initial-value theorem-

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \tag{4-11}$$

7. Final-value theorem-

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{4-12}$$

8. Complex shifting-

$$\mathcal{L}\left[e^{\mp at} f(t)\right] = F(s \pm a) \tag{4-13}$$

9. Real convolution (Complex multiplication)-

$$F_1(s)F_2(s) = \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = \mathcal{L}\left[\int_0^t f_2(\tau)f_1(t-\tau)d\tau\right] = \mathcal{L}[f_1(t) * f_2(t)] \tag{4-14}$$

where $*$ denotes complex convolution.

Inverse Laplace Transformation:

The operation of obtaining f(t) from the Laplace transform F(s) is called the inverse Laplace transformation. The inverse Laplace transform of F(s) is expressed as follows

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds\,, \tag{4-15}$$

where c is a real constant that is greater than the real parts of all the singularities of F(s). Equation (4-15) represents a line integral that is to be evaluated in the s-plane. However, for most engineering purposes the inverse Laplace transform operation can be done simply by referring to the Laplace transform table, such as the one given in Table 4-1. Before using Table 4-1, one may need to do partial-fraction expansion first, then, use the equations in table 4-1.

Partial-fraction expansion when all the poles(*) of the transfer function are simple and real:

Example 4-3: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

$$G(s) = \frac{s+3}{(s+1)(s+2)}$$

Solution:

$$G(s) = \frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where,

$$A = (s+1)G(s)\big|_{s=-1} = \left.\frac{s+3}{s+2}\right|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

and

$$B = (s+2)G(s)\big|_{s=-2} = \left.\frac{s+3}{s+1}\right|_{s=-2} = \frac{-2+3}{-2+1} = -1$$

Therefore,

$$G(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

The inverse Laplace transform of the given transfer function can be obtained by using some equation in Table 4-1, namely,

$$g(t) = \mathcal{L}^{-1}\left(\frac{2}{s+1} + \frac{-1}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{-1}{s+2}\right) = 2e^{-t} - 1e^{-2t} \qquad t \ge 0$$

♦

Note (*): A pole is a value of s that makes a function, such a G(s), infinite, by making the denominator of the function to zero.

Partial-fraction expansion when some poles of the transfer function are of multiple order:
Example 4-4: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

$$G(s)=\frac{1}{s^2(s+1)}$$

Solution:

$$G(s)=\frac{1}{s^2(s+1)}=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{B}{s+1}$$

where,

$$A_2=\left[s^2G(s)\right]\Big|_{s=0}=\left[\frac{1}{s+1}\right]\Bigg|_{s=0}=1$$

$$A_1=\frac{d}{ds}\left[s^2G(s)\right]\Big|_{s=0}=\frac{d}{ds}\left[\frac{1}{s+1}\right]\Bigg|_{s=0}=\left[\frac{-1}{(s+1)^2}\right]\Bigg|_{s=0}=-1$$

and

$$B=(s+1)G(s)\big|_{s=-1}=\frac{1}{s^2}\Bigg|_{s=-1}=1$$

Therefore,

$$G(s)=\frac{1}{s^2(s+1)}=\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}$$

The inverse Laplace of the given transfer function can be obtained by using equations in Table 4-1, namely,

$$g(t)=\mathcal{L}^{-1}\left(\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right)=\mathcal{L}^{-1}\left(\frac{-1}{s}\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)+\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)=-1+t+e^{-t} \qquad t\ge 0$$

♦

Example 4-5: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

$$G(s)=\frac{1}{s(s+1)^3(s+2)}$$

Solution:

$$G(s)=\frac{1}{s(s+1)^3(s+2)}=\frac{A_1}{s+1}+\frac{A_2}{(s+1)^2}+\frac{A_3}{(s+1)^3}+\frac{B}{s+2}+\frac{C}{s}$$

where

$$A_3 = \left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \left[\frac{1}{s(s+2)}\right]\Big|_{s=-1} = -1$$

$$A_2 = \frac{d}{ds}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1} = \left[\frac{-(2s+2)}{s^2(s+2)^2}\right]\Big|_{s=-1} = 0$$

$$A_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[\frac{-2(s+1)}{s^2(s+2)^2}\right]\Big|_{s=-1}$$

$$= \left[\frac{-1}{s^2(s+2)^2} + \frac{2(s+1)}{s^2(s+2)^3} + \frac{2(s+1)}{s^3(s+2)^2}\right]\Big|_{s=-1} = -1$$

$$B = (s+2)G(s)\big|_{s=-2} = \frac{1}{s(s+1)^3}\Big|_{s=-2} = \frac{1}{-2(-2+1)} = 0.5$$

and

$$C = sG(s)\big|_{s=0} = \frac{1}{(s+1)^3(s+2)}\Big|_{s=0} = \frac{1}{(0+1)^3(0+2)} = 0.5$$

Therefore,

$$G(s) = \frac{1}{s(s+1)^3(s+2)} = \frac{-1}{s+1} + \frac{0}{(s+1)^2} + \frac{-1}{(s+1)^3} + \frac{0.5}{s+2} + \frac{0.5}{s}$$

The inverse Laplace of the given transfer function can be obtained by using Table 4-1, namely,

$$g(t) = \mathcal{L}^{-1}\left(\frac{-1}{s+1} + \frac{-1}{(s+1)^3} + \frac{0.5}{s+2} + \frac{0.5}{s}\right) = \mathcal{L}^{-1}\left(\frac{-1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{-1}{(s+1)^3}\right) + \mathcal{L}^{-1}\left(\frac{0.5}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{0.5}{s}\right)$$

$$= 0.5 - e^{-t} + 0.5e^{-2t} - \frac{1}{(3-1)!}t^{3-1}e^{-t} \qquad t \ge 0$$

$$= 0.5 - e^{-t} + 0.5e^{-2t} - 0.5t^2 e^{-t} \qquad t \ge 0$$

♦

Partial-fraction expansion with simple complex-conjugate poles:

Example 4-6: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

$$G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Solution:

$$G(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{s(s+\alpha-j\omega)(s+\alpha+j\omega)}=\frac{A_1}{s+\alpha-j\omega}+\frac{A_2}{s+\alpha+j\omega}+\frac{B}{s}$$

and

$$\alpha=\zeta\omega_n\,;\ \omega=\omega_n\sqrt{1-\zeta^2}$$

where

$$A_1=(s+\alpha-j\omega)G(s)\big|_{s=-\alpha+j\omega}=\left.\frac{\omega_n^2}{s(s+\alpha+j\omega)}\right|_{s=-\alpha+j\omega}=\frac{\omega_n^2}{(-\alpha+j\omega)(2j\omega)}=\frac{\omega_n}{2\omega}e^{-j\left(\theta+\frac{\pi}{2}\right)}$$

$$A_2=(s+\alpha+j\omega)G(s)\big|_{s=-\alpha-j\omega}=\left.\frac{\omega_n^2}{s(s+\alpha-j\omega)}\right|_{s=-\alpha-j\omega}=\frac{\omega_n^2}{(-\alpha-j\omega)(-2j\omega)}=\frac{\omega_n}{2\omega}e^{j\left(\theta+\frac{\pi}{2}\right)}$$

with $\theta=\tan^{-1}\left[\frac{\omega}{-\alpha}\right]$ (please note this angle is in the second quadrant)

and

$$B=sG(s)\big|_{s=0}=\left.\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}\right|_{s=0}=1$$

Therefore,

$$G(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{\omega_n}{2\omega}\left[\frac{e^{-j\left(\theta+\frac{\pi}{2}\right)}}{s+\alpha-j\omega}+\frac{e^{j\left(\theta+\frac{\pi}{2}\right)}}{s+\alpha+j\omega}\right]+\frac{1}{s}$$

The inverse Laplace of the given transfer function can be obtained by using Table 4-1, namely,

$$g(t)=1+\frac{\omega_n}{2\omega}\left[e^{-j\left(\theta+\frac{\pi}{2}\right)}e^{(-\alpha+j\omega)t}+e^{j\left(\theta+\frac{\pi}{2}\right)}e^{(-\alpha-j\omega)t}\right]=1+\frac{\omega_n}{2\omega}e^{-\alpha t}\left[e^{-j\left(\theta+\frac{\pi}{2}-\omega t\right)}+e^{j\left(\theta+\frac{\pi}{2}-\omega t\right)}\right]$$

$$=1+\frac{\omega_n}{\omega}e^{-\alpha t}\left[\frac{e^{j\left(\omega t-\theta-\frac{\pi}{2}\right)}+e^{-j\left(\omega t-\theta-\frac{\pi}{2}\right)}}{2}\right]=1+\frac{\omega_n}{\omega}e^{-\alpha t}\left[\frac{e^{j(\omega t-\theta)}-e^{-j(\omega t-\theta)}}{2j}\right]=1+\frac{\omega_n}{\omega}e^{-\alpha t}\sin(\omega t-\theta)$$

$$=1+\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t-\theta\right)\qquad t\geq 0$$

where

$$\theta=\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{-\zeta}\right]$$ ♦

Laplace Transform Table:

Table 4-1: Laplace Transform Table.

Laplace Transform, F(s)	Time Function, f(t)
1	$\delta(t)$ (Impulse at t = 0)
$\frac{1}{s}$	$u_s(t)$ (Unit Step at t = 0)
$\frac{1}{s^2}$	t (or t [u(t)], Ramp at t = 0)
$\frac{n!}{s^{n+1}}$	t^n (n = positive integer)
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at}-e^{-bt}}{b-a}$
$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin\omega_n t$
$\frac{s}{s^2+\omega_n^2}$	$\cos\omega_n t$
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{n!}{(s+a)^{n+1}}$	$t^n e^{-at}$
$\frac{\omega_n}{(s+a)^2+\omega_n^2}$	$e^{-at}\sin\omega_n t$
$\frac{s+a}{(s+a)^2+\omega_n^2}$	$e^{-at}\cos\omega_n t$

$\frac{1}{(1+sT)^n}$	$\frac{1}{T^n(n-1)!}t^{n-1}e^{-\frac{t}{T}}$
$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$
$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$1+\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t-\phi\right)$ where $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta}$
$\frac{s\,\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$ where $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta}$
$\frac{1}{s(1+sT)}$	$1-e^{-\frac{t}{T}}$
$\frac{1}{s(1+sT)^2}$	$1-\frac{t+T}{T}e^{-\frac{t}{T}}$
$\frac{1}{s^2(1+sT)^2}$	$t-2T+(t+2T)e^{-\frac{t}{T}}$
$\frac{1+as}{s^2(1+sT)}$	$t+(a-T)\left(1-e^{-\frac{t}{T}}\right)$
$\frac{s}{(s^2+\omega_n^2)^2}$	$\frac{1}{2\omega_n}t\sin\omega_n t$
$\frac{s}{(s^2+\omega_{n1}^2)^2(s^2+\omega_{n2}^2)^2}$	$\frac{1}{\omega_{n2}^2-\omega_{n1}^2}(\cos\omega_{n1}t-\cos\omega_{n2}t)$
$\frac{s^2}{(s^2+\omega_n^2)^2}$	$\frac{1}{2\omega_n}(\sin\omega_n t+\omega_n t\cos\omega_n t)$

V. Time-Domain Analysis:

The time response of a control system is usually divided into two parts: the transient response and the steady-state response. Let $c(t)$ denote a time response and be expressed as follows:

$$c(t) = c_t(t) + c_{ss}(t), \tag{5-1}$$

where $c_t(t)$ is the transient response and $c_{ss}(t)$ is the steady-state response.

Typical Test Signals:

To facilitate the time-domain analysis, the following input test signals are often used.

Step Function: The step function represents an instantaneous change in the reference input variable. At time $t < 0$, the signal is zero while at $t \geq 0$, the signal is R, namely,

$$r(t) = \begin{cases} R & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{5-2}$$

where R is a constant. Or,

$$r(t) = R u_s(t), \tag{5-3}$$

where $u_s(t)$ is the unit step function. The step function as a function of time is shown in Figure 5-1.

Fig. 5-1. Step function input.

Ramp Function: The ramp function is a signal to have a constant change in value at the same rate with respect to time and can be expressed as

$$r(t) = \begin{cases} Rt & t \geq 0 \\ 0 & t < 0 \end{cases}. \tag{5-4}$$

As one can see, the slope of the signal is a constant, R. The ramp function is shown in Figure 5-2.

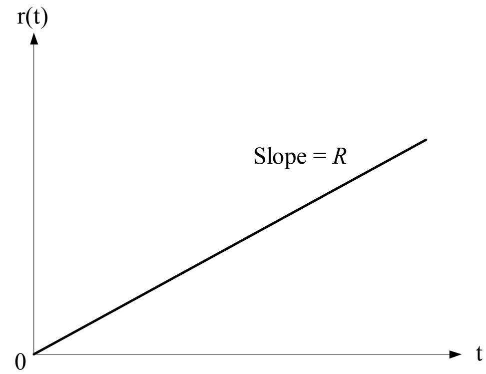


Fig. 5-2. Ramp function input.

Parabolic Function: The mathematical representation of a parabolic function is expressed as follows:

$$r(t) = \begin{cases} \frac{R}{2} t^2 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (5\text{-}5)$$

The graphical representation of the parabolic function is shown in Figure 5-3.

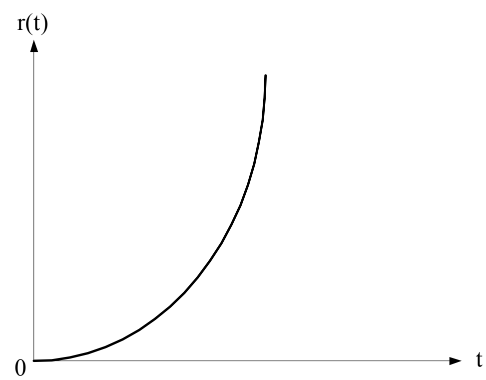


Fig. 5-3. Parabolic function input.

Steady-State Error of Linear Systems:

A closed-loop system with a negative feedback is shown in Figure 5-4. The output can be expressed as a function of the input as follows:

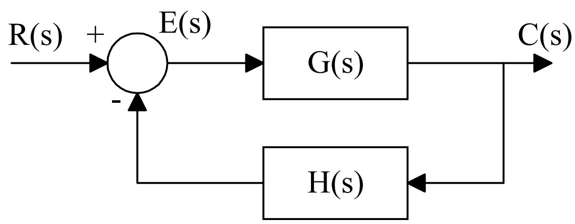


Fig. 5-4. A closed-loop system with a negative feedback.

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s). \tag{5-6}$$

Also, the error, E(s), can be expressed as:

$$E(s) = R(s) - H(s)C(s), \tag{5-7}$$

then,

$$E(s) = \left[1 - \frac{G(s)H(s)}{1 + G(s)H(s)}\right] R(s) = \frac{1}{1 + G(s)H(s)} \cdot R(s). \tag{5-8}$$

The steady-state error in time domain can be obtained by applying the final-value theorem, namely,

$$e_{ss}(t) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot R(s). \tag{5-9}$$

Typically, the inputs for steady-state error considerations are:

(1) If the input is a step function, u(t), of magnitude *R*, or R/s in Laplace transform. Then, the steady-state error of the system can be denoted as

$$e_{ss}(t) = \lim_{s\to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot R(s) = \lim_{s\to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s} = \lim_{s\to 0} \frac{R}{1 + G(s)H(s)}. \tag{5-10}$$

If one defines

$$K_p = \lim_{s\to 0} G(s)H(s), \tag{5-11}$$

where K_p is the step error constant (or position error constant). Then, the equation (5-10) can be simplified as

$$e_{ss}(t) = \lim_{s\to 0} \frac{R}{1 + G(s)H(s)} = \frac{R}{1 + \lim_{s\to 0} G(s)H(s)} = \frac{R}{1 + K_p}. \tag{5-12}$$

One can easily realize that to have the steady-state error to be zero when the input is a step function, K_p must be infinite. The steady-state error due to a unit step function input is also called the steady-state position error.

(2) If the input is a ramp function, its Laplace transform can be expressed as

$$R(s) = \frac{R}{s^2}. \tag{5-13}$$

Then, the steady-state error of the system is

$$e_{ss}(t) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s^2} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)} = \frac{R}{\lim_{s \to 0} s\, G(s)H(s)}. \tag{5-14}$$

If one defines

$$K_v = \lim_{s \to 0} s\, G(s)H(s), \tag{5-15}$$

where K_v is the ramp error constant (or velocity error constant). Therefore, the equation (5-14) can be simplified as

$$e_{ss}(t) = \frac{R}{K_v}. \tag{5-16}$$

Therefore, to have zero steady-state error, K_v needs to be infinite. The steady-state error due to a unit ramp function input is called the steady-state velocity error.

(3) If the input is a parabolic input, its transfer function is listed below

$$R(s) = \frac{R}{s^3}. \tag{5-17}$$

The steady-state error of the system can be obtained as follows

$$e_{ss}(t) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s^3} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)H(s)} = \frac{R}{\lim_{s \to 0} s^2\, G(s)H(s)}. \tag{5-18}$$

If one defines the parabolic error constant (or acceleration error constant),

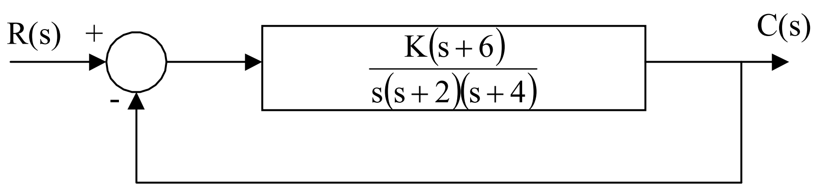
$$K_a = \lim_{s \to 0} s^2\, G(s)H(s). \tag{5-19}$$

Then, the steady-state error due to a parabolic input can be written as

$$e_{ss}(t) = \frac{R}{K_a}, \tag{5-20}$$

and it is also called the steady-state acceleration error.

Example 5-1: A closed-loop system is shown below.



(A) For K = 10, what is the steady-state error to a unit step input?
(B) For K = 10, what is the steady-state error to a unit ramp input?
(C) For K = 10, what is the steady-state error to a unit parabolic input?
(D) Find the value of K such that the steady-state error to a unit ramp input is 5%.

Solution:

(A) For a unit step input and K = 10:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\left[\frac{10(s+6)}{s(s+2)(s+4)}\right]\cdot 1 = \infty .$$

$$e_{ss}(t) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0 .$$

(B) For a unit ramp input and K = 10:

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s\left[\frac{10(s+6)}{s(s+2)(s+4)}\right]\cdot 1 = \lim_{s\to 0}\frac{10(s+6)}{(s+2)(s+4)} = \frac{60}{8} = 7.5$$

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{7.5} = 0.1333 = 13.33\% .$$

(C) For a Parabolic input and K = 10:

$$K_a = \lim_{s\to 0} s^2\,G(s)H(s) = \lim_{s\to 0} s^2\left[\frac{10(s+6)}{s(s+2)(s+4)}\right]\cdot 1 = \lim_{s\to 0} s\cdot\frac{10(s+6)}{(s+2)(s+4)} = 0 .$$

$$e_{ss}(t) = \frac{1}{K_a} = \frac{1}{0} = \infty .$$

(D) From part (B), for a unit ramp input,

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s\left[\frac{K(s+6)}{s(s+2)(s+4)}\right]\cdot 1 = \lim_{s\to 0}\frac{K(s+6)}{(s+2)(s+4)} = \frac{6K}{8}$$

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{6K/8} = 5\% = 0.05 .$$

$$K = \frac{8}{6\cdot(0.05)} = \frac{80}{3} .$$

♦

Type of Feedback Control System:

A stable system can be classified according to the degree of the polynomial for which the error is a constant, and the classification is called the system type. In general, the open-loop transfer function G(s)H(s) can be written as

$$G(s)H(s) = \frac{K(1+a_1 s)(1+a_2 s)\bullet\bullet\bullet(1+a_m s)}{s^T(1+b_1 s)(1+b_2 s)\bullet\bullet\bullet(1+b_n s)}, \qquad (5\text{-}21)$$

where that K, all a's and b's are constants and the T = 0, 1, 2,….. The type of feedback control system is referring to the order of the pole of G(s)H(s) at s = 0. Therefore, the system is of type T. For instance, a feedback control system with

$$G(s)H(s) = \frac{5(1+2s)}{s(1+3s)(1+4s)}$$

is of type 1.

The significance of system types is as follows (when restricted to unity feedback systems):

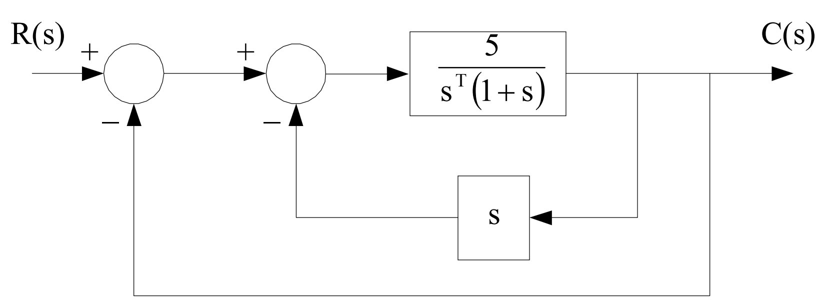
Type 0: System has a non-zero steady-state error to a step input.
Type 1: System has zero steady-state error to a step input. It has a non-zero steady-state error to a ramp input.
Type 2: System has zero steady-state error to both a step input and a ramp input. It has a non-zero steady-state error to a parabolic input.

The following summary of the Steady-State Errors due to Step, Ramp and Parabolic inputs can be constructed, as shown in Table 5-1, where T denotes the type of system.

Table 5-1: Steady-State Error analysis table.

Type	K_p	K_v	K_a	Step	Ramp	Parabolic (Acceleration)
$T = 0$	K	0	0	$e_{ss} = \frac{R}{1+K_p}$	$e_{ss} = \infty$	$e_{ss} = \infty$
$T = 1$	∞	K	0	$e_{ss} = 0$	$e_{ss} = \frac{R}{K_v}$	$e_{ss} = \infty$
$T = 2$	∞	∞	K	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{R}{K_a}$
$T \geq 3$	∞	∞	∞	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = 0$

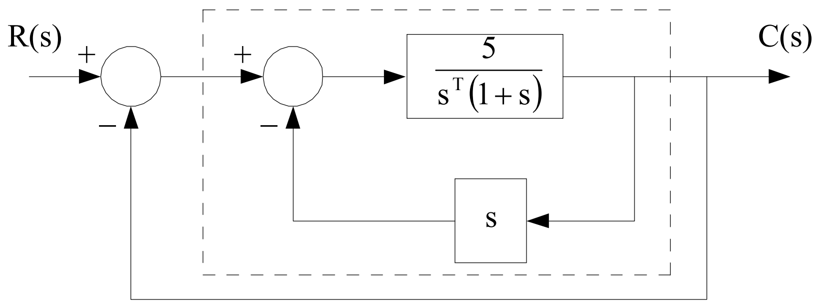
Example 5-2. For the given system with its block diagram listed below:



(A) Find the overall closed-loop transfer function.
(B) If T = 0, find the steady-state position error and velocity error.
(C) If T = 1, find the steady-state position error and velocity error.

Solution:

(A) One can obtain the overall transfer function by applying rules of simplifying block diagram.



One may first apply case 3 in Table 1-1 to obtain the transfer function for the inner closed-loop as

$$G(s)=\frac{\frac{5}{s^T(1+s)}}{1+\frac{5}{s^T(1+s)}\cdot s}=\frac{5}{s^T(1+s)+5s}.$$

Then, use case 3 once again with a unity feedback (H(s)=1), therefore, the overall transfer function can be obtained as

$$\frac{C(s)}{R(s)}=\frac{\frac{5}{s^T(1+s)+5s}}{1+\frac{5}{s^T(1+s)+5s}\cdot 1}=\frac{5}{s^T(1+s)+5s+5}.$$

Or, one can use Mason Rule and get the overall transfer function as

$$\frac{C(s)}{R(s)}=\frac{\frac{5}{s^T(1+s)}}{1+\frac{5}{s^T(1+s)}\cdot s+\frac{5}{s^T(1+s)}\cdot 1}=\frac{5}{s^T(1+s)+5s+5}.$$

(B) T = 0, the open-loop transfer function G(s)H(s) is

$$G(s)H(s)=\frac{5}{(1+s)+5s}=\frac{5}{1+6s}.$$

To find the steady-state position error, the input is assumed as a unit step input,

$R(s)=\frac{1}{s}$, and the position error constant can be calculated as

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\left[\frac{5}{1+6s}\right]\cdot 1 = 5.$$

Therefore, the steady-state position error can be obtained, namely,

$$e_{ss,p}(t) = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}.$$

To find the steady-state velocity error, the input is assumed as a unit ramp input, $R(s) = \frac{1}{s^2}$, and the velocity error constant can be calculated as

$$K_v = \lim_{s\to 0} s\, G(s)H(s) = \lim_{s\to 0} s \cdot \frac{5}{1+6s} = 0.$$

Therefore, the steady-state velocity error is

$$e_{ss,v}(t) = \frac{1}{K_v} = \frac{1}{0} = \infty.$$

(C) T = 1, the open-loop transfer function G(s)H(s) is

$$G(s)H(s) = \frac{5}{s(1+s)+5s} = \frac{5}{s(s+6)}.$$

To find the steady-state position error, the input is assumed as a unit step input, $R(s) = \frac{1}{s}$, and the position error constant can be calculated as

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\left[\frac{5}{s(s+6)}\right]\cdot 1 = \infty.$$

Therefore, the steady-state position error can be obtained, namely,

$$e_{ss,p}(t) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0.$$

To find the steady-state velocity error, the input is assumed as a unit ramp input, $R(s) = \frac{1}{s^2}$, and the velocity error constant can be calculated as

$$K_v = \lim_{s\to 0} s\, G(s)H(s) = \lim_{s\to 0} s \cdot \frac{5}{s(s+6)} = \lim_{s\to 0} \frac{5}{s+6} = \frac{5}{6}.$$

Therefore, the steady-state velocity error is

$$e_{ss,v}(t) = \frac{1}{K_v} = \frac{1}{5/6} = \frac{6}{5}.$$ ♦

First-Order Systems:

The transfer function of a first-order system can be expressed as

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s\tau + 1}, \tag{5-22}$$

and, the output of the system can be written as

$$C(s) = \frac{1}{s\tau + 1} \cdot R(s). \tag{5-23}$$

The unit step response of the system in time domain can be obtained by inverse of the Laplace transform, or using the formula listed in Table 4-1, namely,

$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s\tau + 1} \cdot \frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-1}{s + \frac{1}{\tau}}\right] = 1 - e^{\frac{-t}{\tau}} \qquad t \geq 0, \tag{5-24}$$

where τ is the time constant of the given control system. By plotting the output signal in time domain, Figure 5-5 can be obtained.

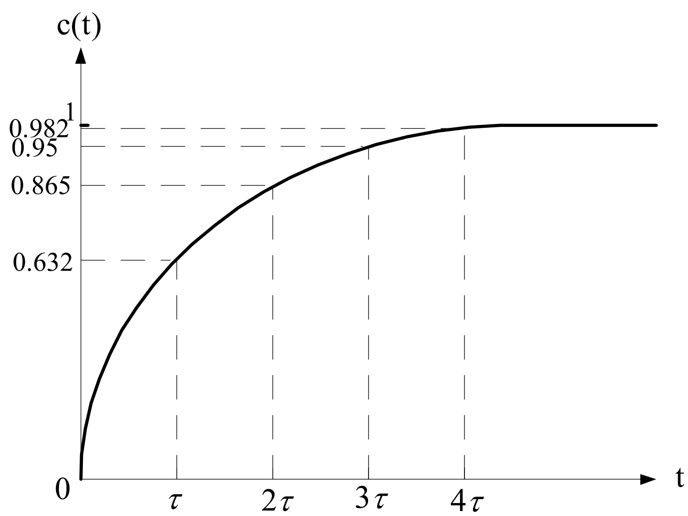


Fig. 5-5. Typical unit step input response of a first-order system.

Classical Second-Order Systems:

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for understanding of design and analysis techniques. The transfer function of a classical second-order system can be represented as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{5-25}$$

where

$$\zeta = \text{damping ratio} \qquad \begin{aligned} &\zeta > 1: \text{overdamped,} \\ &\zeta = 1: \text{critically damped,} \\ &\zeta < 1: \text{underdamped,} \end{aligned}$$

and

ω_n = undamped natural frequency.

When $\zeta < 1$ and a unit step is injected into the system,

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right), \tag{5-26}$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right). \tag{5-27}$$

The time domain plot for such a second-order system with a unit step input is shown in Figure 5-6.

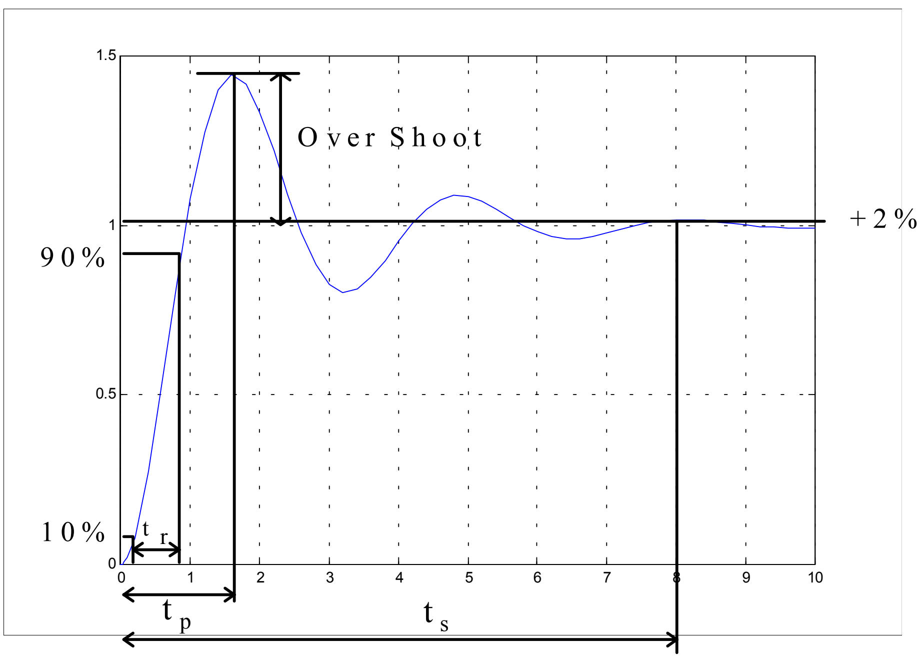


Fig. 5-6. The typical unit step input response of a second-order system.

There are some important performance criteria that are typically used to characterize the transient response to a unit step input include percent (maximum) overshoot, peak time, rise time and settling time.

Percent overshoot: the percentage of the maximum overshoot (the largest deviation of the output over the step input during the transient state).

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% . \quad (5\text{-}28)$$

Peak time: The time it takes to reach the first peak.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \text{ seconds.} \quad (5\text{-}29)$$

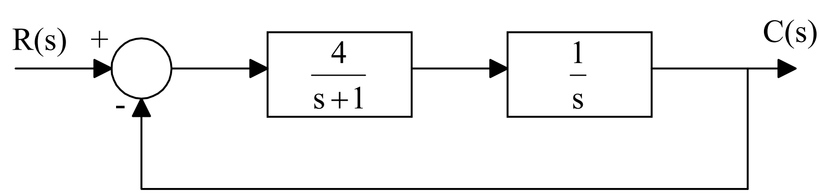
Rise time: The time required for the output to rise from 10% of the final value to 90% of the final values.

$$t_r = \frac{0.8+2.5\zeta}{\omega_n} \text{ seconds.} \quad (5\text{-}30)$$

Settling time: The time it takes for the system to settle within certain percentage of the final value.

$$t_s = \frac{4}{\zeta\omega_n} \text{ seconds. (for 2\%)} \quad (5\text{-}31)$$

Example 5-3: The transfer function of the second order system given below is obtained in



Example 1-1 as $\frac{C(s)}{R(s)} = \frac{4}{s^2+s+4}$. What are the percent overshoot, peak time, rise time and the settling time for this system?

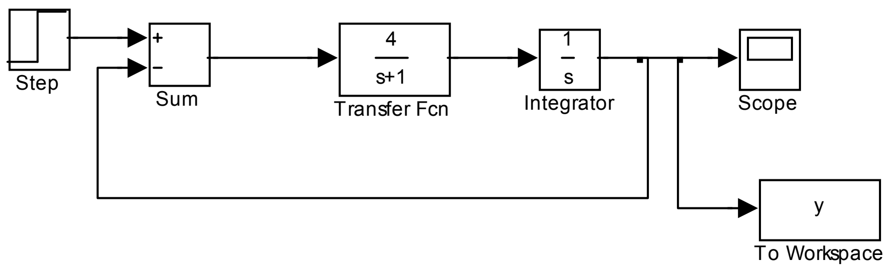
Solution:

By comparing the actual transfer function of the system with the classical second-order equation, namely,

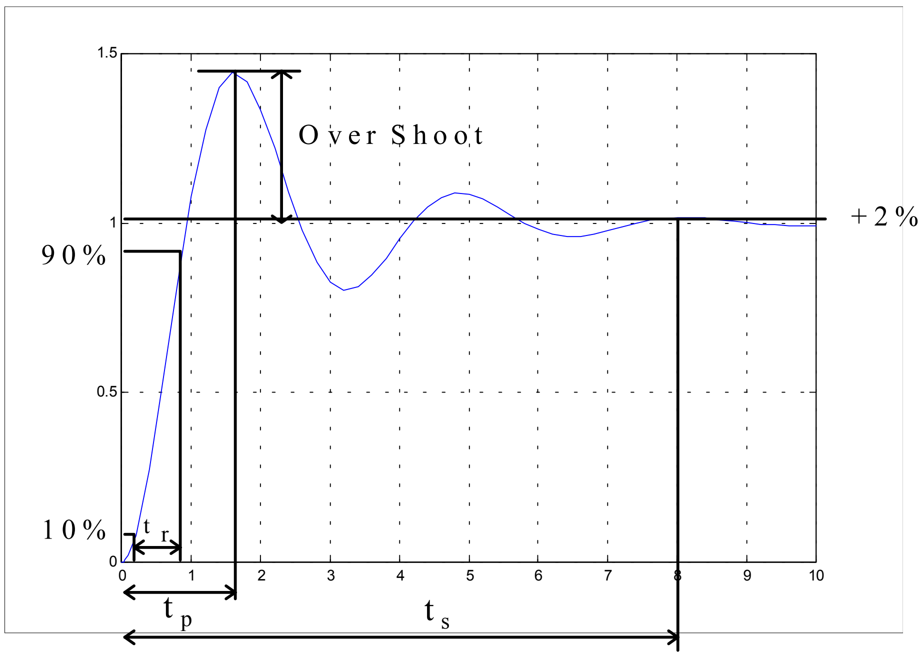
$$\frac{C(s)}{R(s)} = \frac{4}{s^2+s+4} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

One should be able to obtained the undamped natural frequency and the ramping ratio as

$$\omega_n = 2 ; \ \zeta = \frac{1}{2\omega_n} = 0.25$$



$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + s + 4} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



% Overshoot: $\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{-\frac{0.25\pi}{\sqrt{1-0.25^2}}} \times 100 = 44.43\%$

Peak time: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{2\sqrt{1-0.25^2}} = 1.6223 \text{ seconds}$

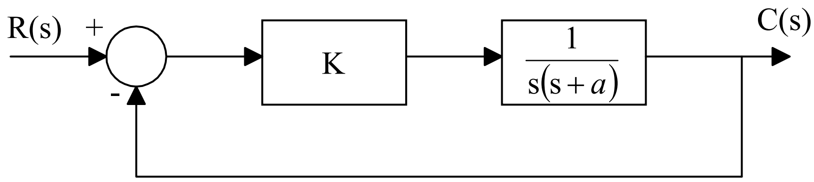
Rise time: $t_r = \frac{0.8 + 2.5\zeta}{\omega_n} = \frac{0.8 + 2.5 \times 0.25}{2} = 0.7125 \text{ second}$

Settling time: $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.25 \times 2} = 8 \text{ seconds} \quad \text{(for 2\%)}$

If one uses Matlab/Simulink to model this simple system, one should be able to get the plot shown above and confirm the answers obtained with equations (5-28)~(5-31).

♦

Example 5-4: Find K and a for the closed-loop system below such that the transient response to a step input satisfies: $\%OS \le 5\%$ & $t_s \le 4$ seconds



Solution:

The transfer function of the system can be obtained as

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+a)}}{1+\frac{K}{s(s+a)}\cdot 1} = \frac{K}{s^2+as+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Therefore,

$$K = \omega_n^2;\ a = 2\zeta\omega_n.$$

The percent overshoot is required to be less than or equal to 5%, hence,

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = 5\%$$

$$\Rightarrow \quad -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.05) = -2.995732$$

$$\Rightarrow \quad (0.95357)\sqrt{1-\zeta^2} = \zeta$$

$$\Rightarrow \quad 0.9093 - 0.9093\zeta^2 = \zeta^2$$

$$\Rightarrow \quad \zeta^2 = \frac{0.9093}{1.9093}$$

$$\Rightarrow \quad \zeta = \sqrt{\frac{0.9093}{1.9093}} = \sqrt{0.47625} = 0.69$$

To find the undamped natural frequency, one needs to use the other given condition, namely,

$$t_s = \frac{4}{\zeta\omega_n} = 4$$

$$\Rightarrow \quad \zeta\omega_n = 1$$

$$\Rightarrow \quad \omega_n = \frac{1}{\zeta} = \frac{1}{0.69} = 1.4491$$

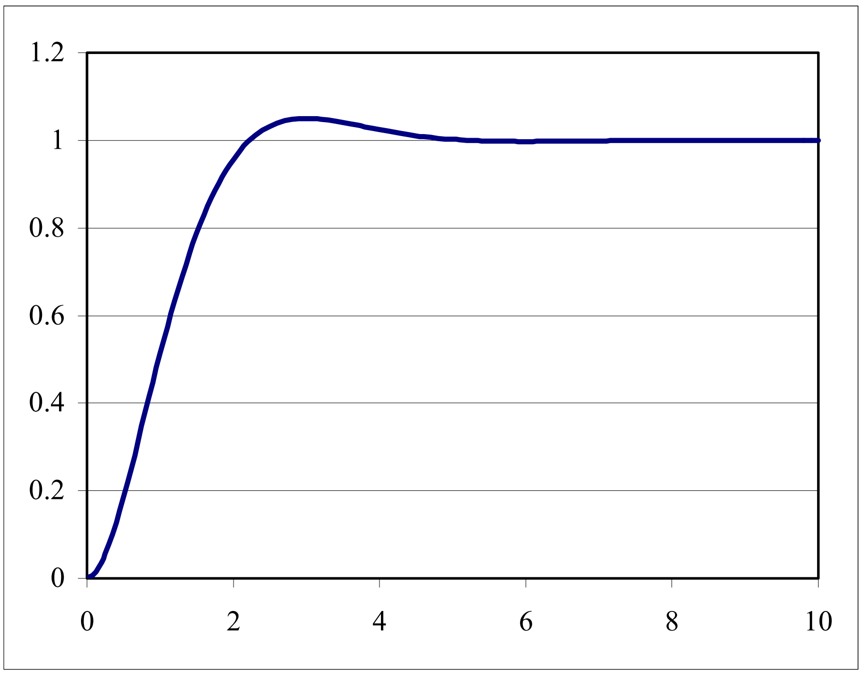
Now, one should be able to find K and a as follows:

$K = \omega_n^2 = 2.1$

and

$a = 2\zeta\omega_n = 2 \cdot 1 = 2$

Other than using Matlab/Simulink to verify the solution, one may use a spreadsheet to plot the output as given by equations (5-26) and (5-27).



♦

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